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A surjectivity result for quasibounded operators

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ABSTRACT

Using a degree theory for countably 1-contractive operators, we show a surjectivity theorem for such quasibounded operators. Moreover, the existence of an eigenvalue for these operators is presented.

1. Introduction

Let f be a continuous operator from a real Banach space E into itself. The problem of finding solutions for an equation of the form

$$x - f(x) = y \quad (y \in E) \quad \text{or} \quad f(x) = \lambda x \quad (\lambda \in \mathbb{R})$$

is one of the most important topics in nonlinear analysis. It was proved in [1] that if f is condensing and strictly quasibounded, then $I - f$ is surjective, where I is the identity operator on E. It is known that there is a 1-set contraction which is not condensing; see e.g. [2]. In order to obtain a surjectivity result for 1-set contractions, an additional condition might be inevitable, such as the closedness of $I - f$, as in [3].

This work deals with the above problem for a large class of operators that includes 1-set contractions, called countably 1-contractive operators. The use of countable sets was initiated by Daher [4] and it turned out in [5] that countability, e.g. a sequence of approximate solutions, could be sufficient for finding solutions of nonlinear differential equations. In view of the applications, Väth [6] established a fixed point index theory for countably condensing operators.

In Section 2, we introduce a degree theory for countably 1-contractive operators which is deduced from the index theory for countably condensing operators, as a degree theory for 1-set contractions based on Nussbaum's index theory [7] was discussed in [3]. In Section 3, we use the degree theory to prove that if f is countably 1-contractive and strictly quasibounded, then the equation $x - f(x) = y$ is solvable for every y, provided that $(I - f)(\overline{B}(0, r))$ is closed for each $r > 0$; see Theorem 1. Moreover, it is shown that under suitable conditions on f and λ the eigenvalue problem $f(x) = \lambda x$ is solvable.

In what follows, E will always be a real Banach space. Given a subset Ω of E, the closure, the boundary, and the convex hull of Ω in E are denoted by $\overline{\Omega}$, $\partial\Omega$, and $\mathrm{co}\,\Omega$, respectively.

A functional $\gamma : \{M \subseteq E : M \text{ is bounded}\} \to [0, \infty)$ is said to be a *measure of noncompactness* on E if it satisfies the following properties:

(1) $\gamma(\overline{M}) = \gamma(M)$;
(2) $\gamma(\mathrm{co}\, M) = \gamma(M)$;

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(3) $\gamma(M) = 0$ if and only if M is relatively compact;
(4) $\gamma(M \cup N) = \max\{\gamma(M), \gamma(N)\}$;
(5) $\gamma(M + N) \leq \gamma(M) + \gamma(N)$; and
(6) $\gamma(\alpha M) = \alpha\gamma(M)$ for every nonnegative real number α.

In this case, (4) implies that $\gamma(N) \leq \gamma(M)$ if $N \subseteq M$. Note that the Kuratowski or the Hausdorff measure of noncompactness has the above properties; see [2].

Let Ω be a subset of E and γ a measure of noncompactness on E, and $k \geq 0$. A continuous operator $f : \Omega \to E$ is said to be:

(1) *countably k-contractive* if $\gamma(f(C)) \leq k\gamma(C)$ for each countable bounded set $C \subseteq \Omega$;
(2) *countably condensing* if $\gamma(f(C)) < \gamma(C)$ for each countable bounded set $C \subseteq \Omega$ with $\gamma(C) > 0$;
(3) *k-set-contractive* if $\gamma(f(B)) \leq k\gamma(B)$ for all bounded sets $B \subseteq \Omega$;
(4) *condensing* if $\gamma(f(B)) < \gamma(B)$ for all bounded sets $B \subseteq \Omega$ with $\gamma(B) > 0$.

A continuous homotopy $h : [0, 1] \times \Omega \to E$ is said to be:

(1) *countably 1-contractive* if $\gamma(h([0, 1] \times C)) \leq \gamma(C)$ for each countable bounded set $C \subseteq \Omega$;
(2) *countably condensing* if $\gamma(h([0, 1] \times C)) < \gamma(C)$ for each countable bounded set $C \subseteq \Omega$ with $\gamma(C) > 0$.

Note that every k-set-contractive operator is countably k-contractive and every countably condensing operator is countably 1-contractive.

2. Degree theory

We introduce a degree theory for countably 1-contractive operators which is based on the fixed point theory for countably condensing operators developed in [6], as in [3] for 1-set contractions.

Let Ω be a bounded open subset of a real Banach space $(E, \|\cdot\|)$ and $f : \overline{\Omega} \to E$ a countably 1-contractive operator such that $f(\partial\Omega)$ is bounded and

$$\|(I - f)(x)\| \geq c \quad \text{for all } x \in \partial\Omega \text{ and for some } c > 0. \tag{2.1}$$

If $g : \overline{\Omega} \to E$ is any countably k-contractive operator with $0 \leq k < 1$ such that

$$\|f(x) - g(x)\| < c \quad \text{for all } x \in \partial\Omega, \tag{2.2}$$

then $g(x) \neq x$ for all $x \in \partial\Omega$ and g is obviously countably condensing. In view of the fixed point index for countably condensing operators (see [6]), ind (g, Ω) is well defined and one uses this degree to define the degree of the operator $I - f$ on Ω over f through the relation

$$\deg(I - f, \Omega, 0) = \text{ind}\,(g, \Omega). \tag{2.3}$$

To justify the above definition, we first observe that a countably k-contractive operator $g : \overline{\Omega} \to E$ with $0 \leq k < 1$ for which (2.2) holds always exists. Indeed, it follows from

$$\|f(x)\| \leq m \quad \text{for all } x \in \partial\Omega \text{ and for some } m > 0$$

that for any $k \in (1 - cm^{-1}, 1)$ with $k \geq 0$ the operator $g := kf : \overline{\Omega} \to E$ is countably k-contractive and we have

$$\|f(x) - g(x)\| = (1 - k)\|f(x)\| \leq (1 - k)m < c \quad \text{for all } x \in \partial\Omega.$$

Next, $\deg(I - f, \Omega, 0)$ given by (2.3) is independent of the choice of g. In fact, if $g_1 : \overline{\Omega} \to E$ is another countably k_1-contractive operator with $0 \leq k_1 < 1$ for which (2.2) holds, then the continuous operator $h : [0, 1] \times \overline{\Omega} \to E$ defined by

$$h(t, x) := tg(x) + (1 - t)g_1(x) \quad \text{for } (t, x) \in [0, 1] \times \overline{\Omega}$$

is countably condensing, since for each countable set $C \subseteq \overline{\Omega}$ with $\gamma(C) > 0$ we have

$$\begin{aligned}\gamma\,(h([0, 1] \times C)) &\leq \gamma\,(\mathrm{co}[g(C) \cup g_1(C)]) = \max\{\gamma(g(C)), \gamma(g_1(C))\}\\ &\leq \max\{k, k_1\}\gamma(C) < \gamma(C).\end{aligned}$$

Moreover, we have $h(t, x) \neq x$ for all $(t, x) \in [0, 1] \times \partial\Omega$ because it follows from (2.1) and (2.2) that

$$\begin{aligned}\|x - h(t, x)\| &= \|x - f(x) + t(f(x) - g(x)) + (1 - t)(f(x) - g_1(x))\|\\ &\geq \|x - f(x)\| - t\|f(x) - g(x)\| - (1 - t)\|f(x) - g_1(x)\|\\ &> c - tc - (1 - t)c = 0.\end{aligned}$$

Hence the homotopy invariance of the index for countably condensing operators implies that

$$\text{ind}\,(g, \Omega) = \text{ind}\,(h(1, \cdot), \Omega) = \text{ind}\,(h(0, \cdot), \Omega) = \text{ind}\,(g_1, \Omega).$$

Thus, $\deg(I - f, \Omega, 0)$ is well defined.

The above degree has the following basic properties which will later be needed.

Lemma 1. *Let Ω be a bounded open subset of a Banach space E. Then the following statements hold:*

(1) *(Existence) Let $f : \overline{\Omega} \to E$ be a countably 1-contractive operator such that $f(\overline{\Omega})$ is bounded and $(I - f)(\overline{\Omega})$ is closed. Suppose that there is a real number $c > 0$ such that $\|x - f(x)\| \geq c$ for all $x \in \partial\Omega$. If $\deg(I - f, \Omega, 0) \neq 0$, then f has a fixed point in Ω.*
(2) *(Normalization) If $f \equiv 0$ and $0 \in \Omega$, then $\deg(I - f, \Omega, 0) = 1$.*
(3) *(Homotopy invariance) If $h : [0, 1] \times \overline{\Omega} \to E$ is a countably 1-contractive homotopy such that $h([0, 1] \times \overline{\Omega})$ is bounded and $\|x - h(t, x)\| \geq c$ for all $(t, x) \in [0, 1] \times \partial\Omega$ and for some $c > 0$, then $\deg(I - h(0, \cdot), \Omega, 0) = \deg(I - h(1, \cdot), \Omega, 0)$.*

Proof. (1) For each $n \in \mathbb{N}$, let k_n be a real number such that $0 \leq k_n < 1$ and $k_n \to 1$ as $n \to \infty$ and let $g_n : \overline{\Omega} \to E$ be a countably k_n-contractive operator defined by $g_n(x) := k_n f(x)$ for $x \in \overline{\Omega}$. Set $d_n := \sup\{\|f(x) - g_n(x)\| : x \in \overline{\Omega}\}$. Since $d_n \to 0$ as $n \to \infty$, there is a positive integer N such that $d_n < c$ for all $n \geq N$. It follows from the above definition that

$$\text{ind}\,(g_n, \Omega) = \deg(I - f, \Omega, 0) \neq 0 \quad \text{for all } n \geq N.$$

Hence by the existence property of the index for countably condensing operators, the operator g_n has a fixed point x_n in Ω for all $n \geq N$ which implies $\|x_n - f(x_n)\| \leq d_n$ and therefore $x_n - f(x_n) \to 0$ as $n \to \infty$. Since $(I - f)(\overline{\Omega})$ is closed, there exists an $x_0 \in \overline{\Omega}$ such that $f(x_0) = x_0$. Moreover, by the boundary assumption on f, the point x_0 lies in Ω.

(2) The second statement follows from the fact that $\text{ind}\,(g, \Omega) = 1$ with $g \equiv 0$.

(3) Since $h([0, 1] \times \overline{\Omega})$ is bounded, there is an $m > 0$ such that $\|h(t, x)\| \leq m$ for all $(t, x) \in [0, 1] \times \overline{\Omega}$. Choose a nonnegative real number k so that $1 - (c/2m) < k < 1$ and define a homotopy $H : [0, 1] \times \overline{\Omega} \to E$ by $H(t, x) := kh(t, x)$ for $(t, x) \in [0, 1] \times \overline{\Omega}$. Then for all $(t, x) \in [0, 1] \times \partial\Omega$, we have

$$\|h(t, x) - H(t, x)\| \leq (1 - k)m < \frac{c}{2}$$

and hence

$$\|x - H(t, x)\| \geq \|x - h(t, x)\| - \|h(t, x) - H(t, x)\| \geq \frac{c}{2} > 0.$$

Moreover, H is countably k-contractive because for each countable set $C \subseteq \overline{\Omega}$ we have

$$\gamma(H([0, 1] \times C)) = k\gamma(h([0, 1] \times C)) \leq k\gamma(C).$$

Therefore, the homotopy invariance of the index for countably condensing operators implies that

$$\text{ind}\,(H(0, \cdot), \Omega) = \text{ind}\,(H(1, \cdot), \Omega).$$

Since $\deg(I - h(t, \cdot), \Omega, 0) = \text{ind}\,(H(t, \cdot), \Omega)$ for every $t \in [0, 1]$ by the above definition of our degree, we conclude that

$$\deg(I - h(0, \cdot), \Omega, 0) = \deg(I - h(1, \cdot), \Omega, 0). \quad \square$$

3. Quasibounded operators

We give the following surjectivity theorem for countably 1-contractive operators which includes Corollary 2 of [3] as a special case. For the proof, we take a direct method to use the degree theory, whereas a fixed point theorem is applied in [3].

Theorem 1. *Suppose that $f : E \to E$ is a countably 1-contractive operator such that $f(\overline{B}(0, r))$ is bounded and $(I - f)(\overline{B}(0, r))$ is closed for each $r > 0$, where $B(0, r)$ is an open ball in E with center at 0 and radius r. If f is strictly quasibounded, that is,*

$$\ell = \limsup_{\|x\| \to \infty} \frac{\|f(x)\|}{\|x\|} < 1,$$

then $I - f$ is surjective.

Proof. Fix $y \in E$. By the definition of ℓ, there is a real number $r_0 > 0$ such that

$$\|f(x)\| \leq \ell\|x\| \quad \text{for all } x \in E \text{ with } \|x\| \geq r_0.$$

Choose a real number $r \geq r_0$ such that $(1 - \ell)r > \|y\|$. There are two cases to consider. If $y \in (I - f)(\partial B(0, r))$, there is nothing to prove. Now suppose that $y \notin (I - f)(\partial B(0, r))$. Since $(I - f)(\overline{B}(0, r))$ is closed, there is a real number $\delta > 0$ such that

$$\|x - f(x) - y\| \geq \delta \quad \text{for all } x \in \partial B(0, r).$$

Consider a homotopy $h : [0, 1] \times \overline{B}(0, r) \to E$ defined by

$$h(t, x) := tf(x) + ty \quad \text{for } (t, x) \in [0, 1] \times \overline{B}(0, r).$$

Then $h([0,1] \times \overline{B}(0,r))$ is bounded and h is countably 1-contractive. In fact, for each countable set $C \subseteq \overline{B}(0,r)$, since $h([0,1] \times C) \subseteq \operatorname{co}[(f(C) + \{y\}) \cup \{0\}]$ and f is countably 1-contractive, we have

$$\gamma(h([0,1] \times C)) \leq \gamma(\operatorname{co}[(f(C) + \{y\}) \cup \{0\}]) \leq \gamma(f(C)) + \gamma(\{y\}) \leq \gamma(C).$$

Set $c := \min\{r, \delta, (1-\ell)r - \|y\|\}$. For every $(t,x) \in [0,1] \times \partial B(0,r)$, we obtain

$$\|x - h(t,x)\| \geq c > 0.$$

Indeed, if $t = 0$, then $\|x - h(t,x)\| = \|x\| = r \geq c$. If $t = 1$, then $\|x - h(t,x)\| = \|x - f(x) - y\| \geq \delta \geq c$. If $0 < t < 1$, then $\|x - h(t,x)\| \geq \|x\| - \|f(x)\| - \|y\| \geq (1-\ell)r - \|y\| \geq c$. Lemma 1 now implies that

$$\deg(I - h(1,\cdot), B(0,r), 0) = \deg(I - h(0,\cdot), B(0,r), 0) = 1.$$

Again by Lemma 1, there exists an $x \in B(0,r)$ such that $x = f(x) + y$ or $(I - f)(x) = y$. Thus, in both cases, we conclude that $I - f$ is surjective. □

Remark 1. As mentioned in the introduction, if f is condensing, then the condition that $(I - f)(\overline{B}(0,r))$ is closed for each $r > 0$ is unnecessary; see [1, Theorem 1]. It holds even for countably condensing operators, which can be proved by using the fixed point index theory for such operators.

Using Theorem 1, we prove the solvability of nonlinear equation concerning countably k-contractive operators.

Theorem 2. *Suppose that $f : E \to E$ is a countably k-contractive operator for some $k \geq 0$ such that it is quasibounded, that is,*

$$\ell = \limsup_{\|x\| \to \infty} \frac{\|f(x)\|}{\|x\|} < \infty.$$

Then for every $\lambda > \ell$ with $\lambda \geq k$, the equation $f(x) = \lambda x$ has at least one solution provided that $f(\overline{B}(0,r))$ is bounded and $(\lambda I - f)(\overline{B}(0,r))$ is closed for each $r > 0$.

Proof. Let λ be a real number such that $\lambda > \ell$ and $\lambda \geq k$. Consider an operator $g : E \to E$ defined by $g(x) := (1/\lambda) f(x)$ for $x \in E$. Then g is countably 1-contractive because $\gamma(g(C)) = (1/\lambda)\gamma(f(C)) \leq (k/\lambda)\gamma(C) \leq \gamma(C)$ for each countable bounded set $C \subseteq E$. Note that $g(\overline{B}(0,r))$ is bounded and $(I - g)(\overline{B}(0,r))$ is closed for each $r > 0$ and

$$\limsup_{\|x\| \to \infty} \frac{\|g(x)\|}{\|x\|} = \frac{1}{\lambda} \limsup_{\|x\| \to \infty} \frac{\|f(x)\|}{\|x\|} = \frac{\ell}{\lambda} < 1.$$

Applying Theorem 1, the operator $I - g$ is surjective. Hence there exists an $x_0 \in E$ such that $g(x_0) = x_0$ or $f(x_0) = \lambda x_0$. This completes the proof. □

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